Problem 32.31

Energy density yields the amount of energy per unit volume in a region.

a.) The book approaches the *energy density* of an electric field through the concept of a parallel plate capacitor. The voltage between capacitor plates is "Ed," where "E" is the magnitude of the electric field and "d" the distance between the plates, and the capacitance was derived as $C = \varepsilon_o \frac{A}{d}$, where the "A" term is the area of the plate. The energy wrapped up in a capacitor is $\frac{1}{2}CV^2$.

Putting this all together, we can write:

$$\begin{aligned} U &= \frac{1}{2}CV^2 \\ &= \frac{1}{2}\left(\varepsilon_o \frac{A}{d}\right)(Ed)^2 \\ &= \frac{1}{2}\varepsilon_o Ad(E)^2 \end{aligned}$$

Note that the volume between the plates is "Ad," the amount of energy per unit volume, or energy density, becomes:

$$\begin{aligned} U &= \frac{1}{2}\varepsilon_o (Ad)(E)^2 \\ \Rightarrow \quad \frac{U}{(Ad)} &= \frac{1}{2}\varepsilon_o (E)^2 \\ \Rightarrow \quad \frac{U}{V} &= \frac{1}{2}\varepsilon_o (E)^2 \\ \Rightarrow \quad u_E &= \frac{1}{2}\varepsilon_o (E)^2 \end{aligned}$$

For this problem, that comes out to:

$$\begin{aligned} u_E &= \frac{1}{2}\varepsilon_o (E)^2 \\ &= \frac{1}{2}\left(8.85\text{x}10^{-12}\ \frac{C^2}{N \bullet m^2}\right)\left(100\ \frac{V}{m}\right)^2 \\ &= 44.3\text{x}10^{-9}\ J/m^3 \end{aligned}$$

b.) The book approaches the *energy density* of a magnetic field similarly. The energy wrapped up in a solenoid's magnetic field is $\frac{1}{2}Li^2$. The inductance of a solenoid in terms of its physical parameters is $L = \mu_o n^2 Al$, where "A" is the cross sectional area of the solenoid, "l" is its length and "n" is the *number of winds per unit length*. The relationship between the coil's magnetic field and the current that is generating the field is $B = \mu_o ni$ from which we can deduce that the current is $i = B/\mu_o n$.

Putting this all together, we can write:

$$\begin{aligned} U &= \frac{1}{2}Li^2 \\ &= \frac{1}{2}\left(\mu_o n^2 Al\right)\left(\frac{B}{\mu_o n}\right)^2 \\ &= \frac{1}{2}\frac{B^2}{\mu_o}(Al) \end{aligned}$$

b.) Noticing as before that "Al" is the volume inside the solenoid, we can write the energy per unit volume, or energy density, as:

$$\begin{aligned} U &= \frac{1}{2}\frac{B^2}{\mu_o}(Al) \\ \Rightarrow \quad \frac{U}{(Al)} &= \frac{1}{2}\frac{B^2}{\mu_o} \\ \Rightarrow \quad u_B &= \frac{1}{2}\frac{B^2}{\mu_o} \end{aligned}$$

Putting in the numbers yields:

$$\begin{aligned} u_B &= \frac{1}{2}\frac{B^2}{\mu_o} \\ &= \frac{1}{2}\frac{\left(.5\text{x}10^{-4}\ T\right)^2}{\left(1.26^{-6}\ T \bullet m/A\right)} \\ &= 995\text{x}10^{-6}\ J/m^3 \end{aligned}$$